

On the existence of a local quasi hidden variable (LqHV) model for each N -qudit state and the maximal quantum violation of Bell inequalities

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Abstract

We specify the local quasi hidden variable (LqHV) model reproducing the probabilistic description of all N -partite joint von Neumann measurements on an N -qudit state. Via this local probability model, we derive a new upper bound on the maximal violation by an N -qudit state of N -partite Bell inequalities of any type (either on correlation functions or on joint probabilities) for S observables per site. This new upper bound not only improves for all N , S and d the corresponding results available for general Bell inequalities in the literature but also, for the N -qubit case with two observables per site, reduces exactly to the attainable upper bound known for quantum violations of $2 \times \dots \times 2$ -setting correlation Bell inequalities in a dichotomic case.

Keywords: Local quasi hidden variable (LqHV) modelling; Bell inequalities; the maximal quantum violation

1 Introduction

In 1935, Einstein, Podolsky and Rosen (EPR) argued [1] that *locality* of measurements performed by two parties on perfectly correlated quantum events implies the "simultaneous reality - and thus definite values"¹ of physical quantities described by noncommuting quantum observables. This EPR argument, contradicting the quantum formalism and known as the EPR paradox, seemed to imply a possibility of a *hidden variable* account of quantum measurements.

Analyzing this possibility in 1964-1966, Bell explicitly constructed [2] the hidden variable (HV) model reproducing the statistical properties of all qubit observables. Considering, however, spin measurements of two parties on the two-qubit singlet state, Bell proved [3] that any local hidden variable (LHV) description of these bipartite joint spin measurements on perfectly correlated quantum events disagrees with the statistical predictions of quantum theory. In view of his mathematical results in [2, 3], Bell argued [2, 4] that the EPR paradox should be resolved specifically due to violation of *locality* under multipartite quantum measurements and that "... non-locality is deeply rooted in quantum mechanics itself and will persist in any completion".

However, as we stressed in Sec. 3 of [5], though both specifications of locality, one by EPR in [1] and another by Bell in [3], correspond to the manifestation of the physical principle of

¹See [1], page 778.

local action under multipartite nonsignaling measurements, but – the EPR locality, described in [1] as “without in any way disturbing” systems and measurements at other sites, is a general concept, not in any way associated with the use of some specific mathematical formalism, whereas Bell’s locality (as it is formulated in [3]) constitutes the manifestation of locality specifically in the HV frame. As a result, Bell’s locality implies the EPR locality, but the converse is not true – the EPR locality *does not need* to imply Bell’s locality so that the proved by Bell non-existence of a local HV (LHV) model for the singlet state does not point to resolution of the EPR paradox via violation of the EPR locality. For details, see Sec. 3 in [5].

Nowadays, there is still no a unique conceptual view² on Bell’s concept of quantum nonlocality. However, it is clear that this concept does not mean propagation of interaction faster than light and that it is not equivalent to the concept of quantum entanglement. Moreover, in quantum information, a nonlocal multipartite quantum state is defined purely mathematically - via violation by this state of some Bell inequality³ or, equivalently, via non-existence for each quantum correlation scenario on this state of a local HV (LHV) model.

Note that, from the mathematical point of view, the violation result of Bell in [3] (known in the physical literature as Bell’s theorem) can be conditioned by either of at least two mathematical alternatives: (i) the dependence of a random variable at one site not only on an observable measured at this site but also on measurement settings and outcomes at the other sites; (ii) non-positivity of a scalar measure ν modelling the singlet state. From the physical point of view, a choice between these two mathematical alternatives corresponds to a choice between (i) *nonlocality* and (ii) *nonclassicality*. The latter corresponds to violation of “classical realism” embedded into HV models via probability measures.

Moreover, as we proved by theorem 2 in [11], for the probabilistic description of every quantum correlation scenario, the second alternative does always work. Namely, each quantum correlation scenario admits a *local quasi hidden variable (LqHV) model* – a new general local probability model which we introduced in [11, 12] and where all averages and product expectations of a correlation scenario are reproduced via *local* random variables on a measure space⁴ $(\Omega, \mathcal{F}_\Omega, \mu)$ but, in this triple, a measure μ is real-valued and does not need to be positive⁵. Note that, under the LqHV modelling of a quantum correlation scenario there are no negative probabilities⁶ – though a measure μ can have negative values, all scenario joint probabilities are expressed only via nonnegative values of this measure.

Furthermore, from our recent proof [16] of the existence of a *context-invariant qHV model for all quantum observables and states* on an arbitrary Hilbert space it follows (see proposition 3 in [16]) that *every N -partite quantum state admits a LqHV model*, that is, a single LqHV model for all N -partite joint von Neumann measurements on this state. This new notion is similar by its form to Werner’s [17] notion of a LHV model for an N -partite quantum state but with replacement of a positive measure in the representation for N -partite joint probabilities by a real-valued one. Recall that an arbitrary entangled quantum state does not need to

²See, for example, discussions in [6, 7, 8, 9].

³The general frame for multipartite Bell inequalities for arbitrary numbers of settings and outcomes per site was introduced in [10].

⁴Here, Ω is a set, \mathcal{F}_Ω is an algebra of subsets of Ω and μ is a measure on \mathcal{F}_Ω .

⁵For two canonically conjugate quantum observables, the representation of their averages via the real-valued measure, the *Wigner quasi probability distribution*, was first introduced by Wigner [13].

⁶The introduction of negative probabilities into the quantum formalism was first suggested by Dirac [14] and analysed further by Feynman [15].

admit a LHV model.

All these new results in [11, 12, 16] indicate that, from the point of view of mathematical modelling, the choice of the second above alternative, resulting in the LqHV representations for joint probabilities and product expectations, which are always valid, is much more justified than Bell's choice of the first alternative leading to the conjecture [2, 4] on quantum nonlocality.

Moreover, it is specifically this new type of probabilistic modelling, *the LqHV modelling*, that allowed us to derive [11, 18] the upper bounds on quantum violations of *general*⁷ Bell inequalities which essentially improve the corresponding results available [19, 20, 21, 22] via other mathematical frames, in particular, via the operator space theory in [20, 21, 22].

In the present paper, we analyse further the computational capabilities of the LqHV modelling. Via the LqHV frame, we find a new upper bound on violations by an N -qudit state of general Bell inequalities for S settings per site. This new upper bound incorporates and improves our results in [11, 18]. In the N -qubit case with two observables per site, it reduces exactly to the attainable upper bound known [23] for quantum violations of correlation $2 \times \dots \times 2$ -setting Bell inequalities in a dichotomic case.

The present paper is organized as follows.

In Sec. 2, we present the specific LqHV model for an N -qudit state.

In Sec. 3, we express in the LqHV terms the maximal violation by an N -qudit state of general Bell inequalities and, using the LqHV model introduced in Sec. 2, we find a new upper bound on the maximal violation by an N -qudit state of general Bell inequalities for S settings per site.

In Sec. 4, we compare our new general upper bound with the upper bounds available now in the literature.

In Sec. 5, we formulate the main results and stress the advantages of the LqHV modelling.

2 The LqHV model for an N -qudit state

Denote by \mathfrak{X}_{d^N} the set of all N -qudit observables X on $(\mathbb{C}^d)^{\otimes N}$ and by Λ the set of all real-valued functions

$$\lambda : \mathfrak{X}_{d^N} \rightarrow \bigcup_{X \in \mathfrak{X}_{d^N}} \text{sp} X \quad (1)$$

with values $\lambda(X) \equiv \lambda_X$ in the spectrum $\text{sp} X$ of the corresponding observable X . Let $\pi_{(X_1, \dots, X_m)} : \Lambda \rightarrow \text{sp} X_1 \times \dots \times \text{sp} X_m$ be the canonical projection on Λ :

$$\begin{aligned} \pi_{(X_1, \dots, X_m)}(\lambda) &= (\pi_{X_1}(\lambda), \dots, \pi_{X_m}(\lambda)), \quad m \in \mathbb{N}, \\ \pi_X(\lambda) &= \lambda_X \in \text{sp} X, \end{aligned} \quad (2)$$

and \mathcal{A}_Λ be the algebra of all cylindrical subsets of Λ of the form

$$\begin{aligned} \pi_{(X_1, \dots, X_m)}^{-1}(F) &= \{\lambda \in \Lambda \mid (\pi_{X_1}(\lambda), \dots, \pi_{X_m}(\lambda)) \in F\}, \\ F &\subseteq \text{sp} X_1 \times \dots \times \text{sp} X_m, \end{aligned} \quad (3)$$

for all collections $\{X_1, \dots, X_m\} \subset \mathfrak{X}_{d^N}$, $m \in \mathbb{N}$, of N -qudit observables.

⁷That is, Bell inequalities of any possible type - either on correlation functions, full or restricted, or on joint probabilities or on both. For the definition of a Bell inequality, its general form and specific examples, see [10].

By Proposition 1 and relations (20), (35) in [16], to every N -qudit state $\rho_{d,N}$ on $(\mathbb{C}^d)^{\otimes N}$, there corresponds the unique normalized finitely additive real-valued measure $\mu_{\rho_{d,N}}$ on \mathcal{A}_Λ , defined via the relation

$$\begin{aligned} \mu_{\rho_{d,N}}(\pi_{(X_1, \dots, X_m)}^{-1}(F)) &= \frac{1}{m!} \sum_{(x_1, \dots, x_m) \in F} \text{tr}[\rho_{d,N} \{P_{X_1}(x_1) \cdot \dots \cdot P_{X_m}(x_m)\}_{\text{sym}}], \\ F &\subseteq \text{sp}X_1 \times \dots \times \text{sp}X_m, \end{aligned} \quad (4)$$

for all sets $\pi_{(X_1, \dots, X_m)}^{-1}(F) \in \mathcal{A}_\Lambda$ and all finite collections $\{X_1, \dots, X_m\} \subset \mathfrak{X}_{d,N}$ of N -qudit observables. Here, $P_X(\cdot)$ is the spectral measure of an N -qudit observable X and the notation $\{Z_1 \cdot \dots \cdot Z_m\}_{\text{sym}}$ means the operator sum corresponding to the symmetrization of the operator product $Z_1 \cdot \dots \cdot Z_m$ with respect to all permutations of its factors. From (4) and the spectral theorem it follows

$$\frac{1}{m!} \text{tr}[\rho_{d,N} \{X_1 \cdot \dots \cdot X_m\}_{\text{sym}}] = \int_{\Lambda} \pi_{X_1}(\lambda) \cdot \dots \cdot \pi_{X_m}(\lambda) \mu_{\rho_{d,N}}(d\lambda). \quad (5)$$

For an N -qudit state $\rho_{d,N}$ and arbitrary N -qudit observables of the form

$$\begin{aligned} \tilde{X}_n &= \mathbb{I}_{(\mathbb{C}^d)^{\otimes (n-1)}} \otimes X_n \otimes \mathbb{I}_{(\mathbb{C}^d)^{\otimes (N-n)}}, \\ X_n &\in \mathfrak{X}_d, \quad n = 1, \dots, N, \end{aligned} \quad (6)$$

relations (4), (5) imply the representations⁸

$$\begin{aligned} &\text{tr}[\rho_{d,N} \{P_{X_1}(B_1) \otimes \dots \otimes P_{X_N}(B_N)\}] \\ &= \int_{\Lambda} \chi_{\pi_{\tilde{X}_1}^{-1}(B_1)}(\lambda) \cdot \dots \cdot \chi_{\pi_{\tilde{X}_N}^{-1}(B_N)}(\lambda) \mu_{\rho_{d,N}}(d\lambda), \quad B_n \subseteq \text{sp}X_n, \end{aligned} \quad (7)$$

and

$$\text{tr}[\rho_{d,N} \{X_1 \otimes \dots \otimes X_N\}] = \int_{\Lambda} \pi_{\tilde{X}_1}(\lambda) \cdot \dots \cdot \pi_{\tilde{X}_N}(\lambda) \mu_{\rho_{d,N}}(d\lambda), \quad (8)$$

specified in terms of the measure space $(\Lambda, \mathcal{A}_\Lambda, \mu_{\rho_{d,N}})$ where the normalized measure $\mu_{\rho_{d,N}}$ is real-valued and the random variables $\pi_{\tilde{X}_n}(\lambda)$, $n = 1, \dots, N$, are *local* in the sense that each of them depends only on the corresponding observable X_n at n -th site.

Consider the following general notion introduced in [11, 16].

Definition 1 *An N -partite quantum state ρ admits a local qHV (LqHV) model if, for all observables X_n on each n -th site, all N -partite joint von Neumann probabilities*

$$\text{tr}[\rho \{P_{X_1}(B_1) \otimes \dots \otimes P_{X_N}(B_N)\}], \quad B_n \subseteq \text{sp}X_n, \quad (9)$$

admit the representation

$$\text{tr}[\rho \{P_{X_1}(B_1) \otimes \dots \otimes P_{X_N}(B_N)\}] = \int_{\Omega} P_{X_1}(B_1; \omega) \cdot \dots \cdot P_{X_N}(B_N; \omega) \nu_{\rho}(d\omega) \quad (10)$$

⁸Here, $\chi_A(\lambda)$ is the indicator function of a subset $A \subseteq \Lambda$, i. e. $\chi_A(\lambda) = 1$, if $\lambda \in A$, and $\chi_A(\lambda) = 0$, if $\lambda \notin A$.

in terms of a single measure space $(\Omega, \mathcal{F}_\Omega, \nu_\rho)$ with a normalized real-valued measure ν_ρ and conditional probability distributions $P_{Y_n}(\cdot; \omega)$, $n = 1, \dots, N$, each depending only on the corresponding observable X_n at n -th site.

By this definition, every N -qudit state $\rho_{d,N}$ admits the LqHV model (7), specified by relations (2) - (4).

3 Quantum violations of general Bell inequalities

In this section, we use the LqHV model (7) for finding a new upper bound on the maximal violation by an N -qudit state of *general*⁹ Bell inequalities,.

Consider a correlation scenario, performed on an N -qudit state $\rho_{d,N}$ and with S qudit observables $X_n^{(s)}$, $s = 1, \dots, S$, measured projectively at each n -th site. By restricting the measure $\mu_{\rho_{d,N}}$, given on the algebra \mathcal{A}_Λ of subsets of Λ by relation (4), to the subalgebra of all cylindrical subsets of the form

$$\pi_{(\tilde{X}_1^{(1)}, \dots, \tilde{X}_1^{(S)}, \dots, \tilde{X}_N^{(1)}, \dots, \tilde{X}_N^{(S)})}^{-1}(F), \quad F \subseteq \Omega, \quad (11)$$

where

$$\Omega = \text{sp}X_1^{(1)} \times \dots \times \text{sp}X_1^{(S)} \times \dots \times \text{sp}X_N^{(1)} \times \dots \times \text{sp}X_N^{(S)} \quad (12)$$

and slightly modifying the resulting distribution, we derive for this correlation $S \times \dots \times S$ -setting scenario on a state $\rho_{d,N}$ the following LqHV model:

$$\begin{aligned} & \text{tr}[\rho_{d,N} \{P_{X_1^{(s_1)}}(B_1^{(s_1)}) \otimes \dots \otimes P_{X_N^{(s_N)}}(B_N^{(s_N)})\}] \\ &= \sum_{\omega \in \Omega} \left(\prod_{n=1, \dots, N} \chi_{B_n^{(s_n)}}(x_n^{(s_n)}) \right) \nu_{S \times \dots \times S}^{(\rho_{d,N})}(\omega | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}), \\ & B_n^{(s_n)} \subseteq \text{sp}X_n^{(s_n)}, \quad s_n = 1, \dots, S, \end{aligned} \quad (13)$$

where $\omega = (x_1^{(1)}, \dots, x_1^{(S)}, \dots, x_N^{(1)}, \dots, x_N^{(S)})$ and the normalized real-valued distribution $\nu_{S \times \dots \times S}^{(\rho_{d,N})}$ is specified in Appendix via the N -partite generalization (50) of the bipartite distribution (33).

Let us now use the LqHV model (13) for finding under von Neumann measurements at each site of the maximal violation $\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})}$ by a state $\rho_{d,N}$ of general $S \times \dots \times S$ -setting Bell inequalities. This parameter is defined by relation (51) in [11].

From Eqs. (40)-(42) in [11] it follows that, in the LqHV terms, the maximal Bell violation $\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})}$ takes the form:

$$\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} = \sup_{\substack{X_n^{(s)}, s=1, \dots, S, \\ n=1, \dots, N}} \inf \left\| \tau_{S \times \dots \times S}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}) \right\|_{var}, \quad (14)$$

where: (a) $\tau_{S \times \dots \times S}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)})$ is a real-valued measure in a LqHV model for the correlation scenario on a state ρ_N where S qudit observables $X_n^{(s)}$, $s = 1, \dots, S$,

⁹See footnote 7.

are projectively measured at each n -th site; (b) $\left\| \tau_{S \times \dots \times S}^{(\rho_{d,N})} \right\|_{var}$ is the total variation norm¹⁰ of a measure $\tau_{S \times \dots \times S}^{(\rho_{d,N})}$; (c) infimum is taken over all possible LqHV models for this scenario and supremum – over all collections $X_n^{(s)}$, $s = 1, \dots, S$ of qudit observables measured at each n -th site.

From (14) and the LqHV model (13) it follows

$$\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} \leq \sup_{\substack{X_n^{(s)}, s=1, \dots, S, \\ n=1, \dots, N}} \left\| \nu_{S \times \dots \times S}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}) \right\|_{var}, \quad (15)$$

where

$$\begin{aligned} & \left\| \nu_{S \times \dots \times S}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}) \right\|_{var} \\ &= \sum_{\omega \in \Omega} \left| \nu_{S \times \dots \times S}^{(\rho_{d,N})}(\omega | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}) \right| \end{aligned} \quad (16)$$

is the total variation norm of the real-valued measure $\nu_{S \times \dots \times S}^{(\rho_{d,N})}$ standing in (13). By relation (51), for all states $\rho_{d,N}$, this norm is upper bounded as

$$\begin{aligned} \left\| \nu_{2 \times \dots \times 2}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, X_1^{(2)}, \dots, X_N^{(1)}, X_N^{(2)}) \right\|_{var} &\leq d^{\frac{N-1}{2}}, \quad \text{for } S = 2, \\ \left\| \nu_{S \times \dots \times S}^{(\rho_{d,N})}(\cdot | X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}) \right\|_{var} &\leq d^{\frac{S(N-1)}{2}}, \quad \text{for } S \geq 3. \end{aligned} \quad (17)$$

This and relation (15) imply that, for N -partite joint projective measurements on an N -qudit state $\rho_{d,N}$, the maximal violation $\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})}$ by a state $\rho_{d,N}$ of general Bell inequalities satisfies the relations

$$\begin{aligned} \Upsilon_{2 \times \dots \times 2}^{(\rho_{d,N})} &\leq d^{\frac{N-1}{2}}, \quad \text{for } S = 2, \\ \Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} &\leq d^{\frac{S(N-1)}{2}}, \quad \text{for } S \geq 3, \end{aligned} \quad (18)$$

for all N and d .

Relation (18) and our general upper bound (62) in [11] imply that, under projective parties' measurements at all sites, the maximal violation by an N -qudit state $\rho_{d,N}$ of general Bell inequalities for S settings per site are upper bounded as

$$\Upsilon_{2 \times \dots \times 2}^{(\rho_{d,N})} \leq \min\{d^{\frac{N-1}{2}}, 3^{N-1}\}, \quad \text{for } S = 2, \quad (19)$$

$$\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} \leq \min\{d^{\frac{S(N-1)}{2}}, (2S-1)^{N-1}, (2d)^{N-1} - 2^{N-1} + 1\}, \quad \text{for } S \geq 3,$$

with

$$\begin{aligned} \sup_d \Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} &\leq (2S-1)^{N-1}, \\ \sup_S \Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} &\leq (2d)^{N-1} - 2^{N-1} + 1. \end{aligned} \quad (20)$$

¹⁰On this notion, see [24] and also Sec. 3 in [11].

For $S = d = 2$ and an arbitrary N , the upper bound (19) takes the form

$$\Upsilon_{2 \times \dots \times 2}^{(\rho_{2,N})} \leq 2^{\frac{N-1}{2}}, \quad (21)$$

that is, reduces exactly to the upper bound known [23] for quantum violations of $2 \times \dots \times 2$ -setting Bell inequalities on full correlation functions in a dichotomic case. This bound is attained [23] on the Mermin-Klyshko inequality¹¹ by the generalized Greenberger–Horne–Zeilinger state (GZH).

Therefore, from the new upper bound (19) it follows that, under N -partite joint von Neumann measurements on an N -qubit state, the Mermin-Klyshko inequality gives the maximal violation not only among all correlation $2 \times \dots \times 2$ -setting Bell inequalities, as it was proved in [23], but also among all $2 \times \dots \times 2$ -setting Bell inequalities of any type, either for correlation functions, full or restricted, or for joint probabilities or for both.

The new upper bound (19) on the maximal quantum violation of general N -partite Bell inequalities incorporates and improves our general N -partite upper bounds (62) in [11] and (19) in [18].

In the following section, we also explicitly demonstrate that, for all N , S and d , our upper bounds in [11, 18] and the new upper bound (19) improve all the upper bounds on the maximal quantum violation of general Bell inequalities reported in the literature by other authors [19, 20, 21, 22].

4 Discussion

As it is well-known, in a bipartite case, quantum violations of correlation Bell inequalities cannot exceed [25] the real Grothendieck's constant¹² $K_G^{(R)} = \lim_{n \rightarrow \infty} K_G^{(R)}(n) \in [1.676, 1.783]$ independently on a Hilbert space dimension of a bipartite quantum state and a number S of settings per site.

For the two-qubit singlet state, this upper bound can be more specified in the sense that violations of correlation Bell inequalities by the singlet state are upper bounded [26] by the real Grothendieck's constant $K_G^{(R)}(3) \in [\sqrt{2}, 1.5164]$ of order 3.

However, upper bounds on quantum violations of general Bell inequalities have been much less investigated even for a bipartite case.

Let us now specify the new upper bound (19) on quantum violations of general Bell inequalities for some particular cases and compare these results with the corresponding bounds available in the literature.

4.1 Bipartite case

For a bipartite case, the upper bound (19) reads

$$\begin{aligned} \Upsilon_{2 \times 2}^{(\rho_{d,2})} &\leq \min\{\sqrt{d}, 3\}, \quad \text{for } S = 2, \\ \Upsilon_{S \times S}^{(\rho_{d,2})} &\leq \min\{d^{\frac{S}{2}}, 2 \min\{S, d\} - 1\}, \quad \text{for } S \geq 3, \end{aligned} \quad (22)$$

implying

$$\sup_d \Upsilon_{S \times S}^{(\rho_{d,2})} \leq 2S - 1, \quad \sup_S \Upsilon_{S \times S}^{(\rho_{d,2})} \leq 2d - 1. \quad (23)$$

¹¹On this inequality, see [23] and references therein and also Sec. 3.3 in [10].

¹²The exact value of this constant is not known.

From (22) it follows that, for the two-qubit case ($d = 2$) with two settings per site, the maximal violation of general bipartite Bell inequalities is upper bounded as

$$\Upsilon_{2 \times 2}^{(\rho_{2,2})} \leq \sqrt{2}. \quad (24)$$

Therefore, for the case $S = d = 2$, the new bipartite bound (22) reduces exactly to the attainable upper bound known for quantum violations of Bell inequalities on joint probabilities and correlation functions in a dichotomic case with two settings per site. The latter result is due to the Tsirelson [27] upper bound¹³ $\sqrt{2}$ on quantum violations of the CHSH inequality and the role [28, 23] which the Clauser-Horne (CH) inequality and the Clauser-Horne-Shimony-Holt (CHSH) inequality play in the case $N = S = d = 2$.

The new bounds (22) and (23) incorporate and essentially improve our bipartite upper bounds for general Bell inequalities introduced in [11] by Eq. (65) and in [18] by Eq. (19) (specified for $N = 2$).

Let us now compare our general bipartite bounds with the corresponding results available in the literature.

For all S and d , our general bipartite upper bound (65) in [11] and, hence, the new general bipartite upper bound (22) are essentially better than (i) the upper bound

$$\begin{aligned} \Upsilon_{S \times S}^{(\rho_{2,2})} &\leq 2K_G^{(R)} + 1, & \text{for } d = 2, \\ \Upsilon_{S \times S}^{(\rho_{d,2})} &\leq 2d^2(K_G^{(R)} + 1) - 1, & \text{for } d \geq 3, \end{aligned} \quad (25)$$

on quantum violations of general bipartite Bell inequalities presented by theorem 2 in [19]; (ii) the approximate upper bounds¹⁴

$$\Upsilon_{S \times S}^{(\rho_{d,2})} \preceq \min\{d, S\}, \quad \Upsilon_{S \times S}^{(\rho_{d,2})} \preceq \frac{d}{\ln d}, \quad (26)$$

and the exact upper bound

$$\Upsilon_{S \times S}^{(\rho_{d,2})} \leq 2d \quad (27)$$

on quantum violations of general bipartite Bell inequalities found in [20, 21, 22] via the operator space theory.

4.2 Tripartite case

For a tripartite case, the new upper bound (19) for general Bell inequalities takes the form

$$\begin{aligned} \Upsilon_{2 \times 2 \times 2}^{(\rho_{d,3})} &\leq \min\{d, 9\}, & \text{for } S = 2, \\ \Upsilon_{S \times S \times S}^{(\rho_{d,3})} &\leq \min\{d^S, (2S - 1)^2, 4d^2 - 3\}, & \text{for } S \geq 3, \end{aligned} \quad (28)$$

implying

$$\sup_d \Upsilon_{S \times S \times S}^{(\rho_{d,3})} \leq (2S - 1)^2, \quad \sup_S \Upsilon_{S \times S}^{(\rho_{d,2})} \leq 4d^2 - 3. \quad (29)$$

¹³The Tsirelson upper bound holds for any Hilbert space dimension d .

¹⁴Here, symbol \preceq means an inequality defined up to an unknown universal constant.

For a three-qubit case ($d = 2$), relations (28) reduce to

$$\begin{aligned}\Upsilon_{2 \times 2 \times 2}^{(\rho_{2,3})} &\leq 2, & \Upsilon_{3 \times 3 \times 3}^{(\rho_{2,3})} &\leq 8, \\ \Upsilon_{S \times S \times S}^{(\rho_{2,3})} &\leq 13, & \text{for } S &\geq 4,\end{aligned}\tag{30}$$

where the bound $\Upsilon_{2 \times 2 \times 2}^{(\rho_{2,3})} \leq 2$ is attainable [23] on the tripartite Mermin-Klyshko inequality.

The new general tripartite upper bound (28) incorporates and improves our tripartite upper bounds introduced in [11] by Eq. (67) and in [18] by Eq. (19) specified for $N = 3$.

For all S and d , our general tripartite upper bound (67) in [11] and, hence, the new general tripartite upper bound (28) are better than the upper bound

$$\Upsilon_{S \times S \times S}^{(\rho_{d,3})} \leq 4d^2\tag{31}$$

on quantum violations of general Bell inequalities introduced recently in [22] via the operator space theory.

For a more narrow class of Bell inequalities arising in three-player XOR games, the exact upper bound $\min\{K_G^{(R)}\sqrt{S}, \sqrt{3d}(K_G^{(C)})^{3/2}\}$ was presented in [29]. Here, $K_G^{(C)}$ is the complex Grothendieck's constant.

4.3 N-partite case

As we stressed above in Sec. 3, the new upper bound (19) on quantum violations of general Bell inequalities incorporates and improves our general N -partite upper quantum violation bounds presented by Eq. (62) in [11] and Eq. (19) in [18].

For all S and d , the general upper bound (62) in [11] and, hence, the new general upper bound (19) are essentially better than the general upper bound

$$\Upsilon_{S \times \dots \times S}^{(\rho_{d,N})} \leq (2d)^{(N-1)},\tag{32}$$

presented recently in [22] via the operator space theory.

For the maximal violation by a Schmidt N -partite state of Bell inequalities arising in N -player XOR games, the interesting exact upper bound $2^{\frac{3(N-2)}{2}} K_G^{(C)}$, independent on a Hilbert space dimension d , was presented in [30].

5 Conclusions

In the present paper, we have explicitly demonstrated the computational capabilities of a new type of probabilistic modelling of multipartite joint quantum measurements – the local quasi hidden variable (LqHV) modelling which we introduced and developed in [11, 12, 16, 18].

From the conceptual point of view, the LqHV modelling corresponds just to *nonclassicality* - one of the alternatives, which can explain (see in Introduction) Bell's violation result in [3] but was disregarded by Bell [2, 4] in favor of nonlocality. The choice of the "nonclassicality" alternative results in preserving locality but replacement of "classical realism", embedded into the HV frame via probability measures, by "quantum realism" expressed via real-valued measures in the LqHV frame.

From the mathematical point of view, the local qHV (LqHV) modelling frame is very fruitful for quantum calculations and is valid [11, 16] for the probabilistic description of every

quantum correlation scenario, moreover, of all N -partite joint von Neumann measurements on each N -partite quantum state. It is specifically this new type of probabilistic modelling that allowed us to derive the new upper bound (19) on the maximal quantum violation of general Bell inequalities essentially improving all the upper bounds reported in the literature via other mathematical frames, in particular, the upper bounds [20, 21, 22] found via the operator space theory.

For the N -qubit case with two observables per site, the new general upper bound (19) reduces exactly to the attainable upper bound known [23] for quantum violations of N -partite correlation $2 \times \dots \times 2$ -setting Bell inequalities in a dichotomic case. This proves that, under N -partite joint von Neumann measurements on an N -qubit state, the Mermin-Klyshko inequality gives the maximal violation not only among all $2 \times \dots \times 2$ -setting Bell inequalities on correlation functions but also among $2 \times \dots \times 2$ -setting Bell inequalities of any type, either on correlation functions, full or restricted, or on joint probabilities or on both.

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6 Appendix

Let us first consider a bipartite case ($N = 2$). For simplicity of notations, denote by X_s , $s = 1, \dots, S$, observables measured by Alice and by Y_s , $s = 1, \dots, S$, measured by Bob.

For a bipartite case, the values of the distribution $\nu_{S \times S}^{(\rho_{d,2})}(\cdot \mid X_1, \dots, X_S, Y_1, \dots, Y_S)$, standing in (13), have the form

$$\begin{aligned} & 2\nu_{S \times S}^{(\rho_{d,2})}(x_1, \dots, x_S, y_1, \dots, y_S \mid X_1, \dots, X_S, Y_1, \dots, Y_S) \\ &= \left\{ \prod_{s=1, \dots, S} \alpha_{X_s}^{(+)}(x_s \mid y_1, \dots, y_S) \right\} \text{tr}[\rho_{d,2} \{ \mathbb{I}_{\mathbb{C}^d} \otimes (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(+)} \}] \\ &- \left\{ \prod_{s=1, \dots, S} \alpha_{X_s}^{(-)}(x_s \mid y_1, \dots, y_S) \right\} \text{tr}[\rho_{d,2} \{ \mathbb{I}_{\mathbb{C}^d} \otimes (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(-)} \}], \end{aligned} \quad (33)$$

where (i) the term ”+ h.c.” means the Hermitian conjugate of the previous operator; (ii) notations $Z^{(\pm)}$ mean the positive operators satisfying the relation $Z^{(+)}Z^{(-)} = Z^{(-)}Z^{(+)} = 0$ and decomposing a self-adjoint operator $Z = Z^{(+)} - Z^{(-)}$; (ii) the probability distributions $\alpha_{X_s}^{(\pm)}(\cdot \mid y_1, \dots, y_S)$, $s = 1, \dots, S$, are defined, in view of the Radon-Nikodym theorem [24], by the relation

$$\begin{aligned} & \text{tr}[\rho_{d,2} \{ P_{X_s}(x_s) \otimes (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(\pm)} \}] \\ &= \alpha_{X_s}^{(\pm)}(x_s \mid y_1, \dots, y_S) \text{tr} \left[\rho_{d,2} \left\{ \mathbb{I}_{\mathbb{C}^d} \otimes (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(\pm)} \right\} \right]. \end{aligned} \quad (34)$$

From (34) and the relation

$$\left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var} = \sum_{x_1, \dots, x_S, y_1, \dots, y_S} \left| \nu_{S \times S}^{(\rho_{d,2})}(x_1, \dots, x_S, y_1, \dots, y_S \mid X_1, \dots, X_S, Y_1, \dots, Y_S) \right| \quad (35)$$

it follows that the total variation norm $\left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var}$ of the distribution (33) is upper bounded as

$$\left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var} \leq \frac{1}{2} \sum_{y_1, \dots, y_S} \text{tr} [\tilde{\rho}_{d,2} | P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.} |], \quad (36)$$

where $\tilde{\rho}_{d,2}$ is the qudit state at Bob's site which is reduced from the two-qudit state $\rho_{d,2}$ and

$$\begin{aligned} & | P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.} | \\ &= (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(+)} + (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(-)} \end{aligned} \quad (37)$$

is the absolute value operator.

Calculating operator (37), we find

$$\begin{aligned} & \sum_{y_1, \dots, y_S} | P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.} | \\ &= \sum_{k_1, \dots, k_S} |\beta_{k_1, \dots, k_S}| \{ |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_1}^{(k_1)}| + |\phi_{Y_S}^{(k_S)}\rangle \langle \phi_{Y_S}^{(k_S)}| \\ & \quad + (\frac{\alpha_{k_S k_1} \beta_{(k_1, \dots, k_S)}^2}{|\beta_{(k_1, \dots, k_S)}|^2} |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_S}^{(k_S)}| + \text{h.c.}) \}^{\frac{1}{2}}, \end{aligned} \quad (38)$$

where $\phi_{Y_s}^{(k_s)}$, $k_s = 1, \dots, d$, are the orthonormal eigenvectors of an observable Y_s and

$$\begin{aligned} \alpha_{k_S k_1} &= \langle \phi_{Y_S}^{(k_S)} | \phi_{Y_1}^{(k_1)} \rangle, \\ \beta_{k_1, \dots, k_S} &= \langle \phi_{Y_1}^{(k_1)} | \phi_{Y_2}^{(k_2)} \rangle \langle \phi_{Y_2}^{(k_2)} | \phi_{Y_3}^{(k_3)} \rangle \cdot \dots \cdot \langle \phi_{Y_{S-1}}^{(k_{S-1})} | \phi_{Y_S}^{(k_S)} \rangle. \end{aligned} \quad (39)$$

From (36) - (39) it follows

$$\begin{aligned} \left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var} &\leq \frac{1}{2} \sum_{k_1, \dots, k_S} |\beta_{k_1, \dots, k_S}| \text{tr} [\tilde{\rho}_{d,2} \{ |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_1}^{(k_1)}| + |\phi_{Y_S}^{(k_S)}\rangle \langle \phi_{Y_S}^{(k_S)}| \\ & \quad + (\frac{\alpha_{k_S k_1} \beta_{(k_1, \dots, k_S)}^2}{|\beta_{(k_1, \dots, k_S)}|^2} |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_S}^{(k_S)}| + \text{h.c.}) \}^{\frac{1}{2}}]. \end{aligned} \quad (40)$$

If $S = 2$, then $\beta_{k_1 k_2} = \alpha_{k_2 k_1}^* = \alpha_{k_1 k_2}$ and

$$\begin{aligned} & |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_1}^{(k_1)}| + |\phi_{Y_2}^{(k_2)}\rangle \langle \phi_{Y_2}^{(k_2)}| + (\frac{\alpha_{k_2 k_1} \alpha_{k_1 k_2}^2}{|\alpha_{k_1 k_2}|^2} |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_2}^{(k_2)}| + \text{h.c.}) \\ &= (|\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_1}^{(k_1)}| + |\phi_{Y_2}^{(k_2)}\rangle \langle \phi_{Y_2}^{(k_2)}|)^2. \end{aligned} \quad (41)$$

Taking into account (40), (41) and that sums $\sum_{k_2} |\alpha_{k_1 k_2}|$ and $\sum_{k_1} |\alpha_{k_1 k_2}|$ are upper bounded

by \sqrt{d} , we derive for $S = 2$:

$$\begin{aligned} \left\| \nu_{2 \times 2}^{(\rho_{d,2})} \right\|_{var} &\leq \frac{1}{2} \sum_{k_1, k_2} |\alpha_{k_1 k_2}| \text{tr} [\tilde{\rho}_{d,2} \{ |\phi_{Y_1}^{(k_1)}\rangle \langle \phi_{Y_1}^{(k_1)}| + |\phi_{Y_2}^{(k_2)}\rangle \langle \phi_{Y_2}^{(k_2)}| \}] \\ &\leq \sqrt{d}. \end{aligned} \quad (42)$$

Let now $S \geq 3$. Since $\text{tr}[\rho\sqrt{Z}] \leq \sqrt{\text{tr}[\rho Z]}$ for each ρ and every positive operator Z , relation (40) implies

$$\left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var} \leq \frac{1}{2} \sum_{k_1, \dots, k_S} |\beta_{k_1, \dots, k_S}| \{ \text{tr}[\tilde{\rho}_{d,2}(|\phi_{Y_1}^{(k_1)}\rangle\langle\phi_{Y_1}^{(k_1)}| + |\phi_{Y_S}^{(k_S)}\rangle\langle\phi_{Y_S}^{(k_S)}|)] \quad (43)$$

$$+ 2|\langle\phi_{Y_S}^{(k_S)}|\tilde{\rho}_{d,2}|\phi_{Y_1}^{(k_1)}\rangle| \}^{\frac{1}{2}} \quad (44)$$

Taking into account that

$$|\sum_m \gamma_m \xi_m| \leq \{ \sum_m |\gamma_m|^2 \sum_m |\xi_m|^2 \}^{\frac{1}{2}} \quad (45)$$

for all γ, ξ , and the relations

$$\begin{aligned} \sum_{k_1, \dots, k_S} |\beta_{k_1, \dots, k_S}|^2 &\leq d, \\ \sum_{k_1, \dots, k_S} \text{tr}[\tilde{\rho}_{d,2}(|\phi_{Y_1}^{(k_1)}\rangle\langle\phi_{Y_1}^{(k_1)}| + |\phi_{Y_S}^{(k_S)}\rangle\langle\phi_{Y_S}^{(k_S)}|)] &\leq d^{(S-1)}, \\ \sum_{k_1, \dots, k_S} |\langle\phi_{Y_S}^{(k_S)}|\tilde{\rho}_{d,2}|\phi_{Y_1}^{(k_1)}\rangle| &\leq d^{(S-1)}, \end{aligned} \quad (46)$$

for $S \geq 3$, we finally derive:

$$\left\| \nu_{S \times S}^{(\rho_{d,2})} \right\|_{var} \leq d^{\frac{S}{2}}. \quad (47)$$

For an N -partite case, we use in the LqHV representation (13) the real-valued distribution

$$\nu_{S \times \dots \times S}^{(\rho_{d,N})}(\omega \mid X_1^{(1)}, \dots, X_1^{(S)}, \dots, X_N^{(1)}, \dots, X_N^{(S)}), \quad (48)$$

which is quite similar by its construction to distribution (33) but with the replacement of the terms

$$\mathbb{I}_{\mathbb{C}^d} \otimes \frac{1}{2} (P_{Y_1}(y_1) \cdot \dots \cdot P_{Y_S}(y_S) + \text{h.c.})^{(\pm)} \quad (49)$$

by the N -partite tensor product terms

$$\begin{aligned} &\mathbb{I}_{\mathbb{C}^d} \otimes \left\{ \frac{1}{2} (P_{X_1^{(1)}}(x_1^{(1)}) \cdot \dots \cdot P_{X_1^{(S)}}(x_1^{(S)}) + \text{h.c.}) \right. \\ &\left. \otimes \dots \otimes \frac{1}{2} (P_{X_N^{(1)}}(x_N^{(1)}) \cdot \dots \cdot P_{X_N^{(S)}}(x_N^{(S)}) + \text{h.c.}) \right\}^{(\pm)}. \end{aligned} \quad (50)$$

As a result, we derive

$$\begin{aligned} \left\| \nu_{2 \times \dots \times 2}^{(\rho_{d,N})} \right\|_{var} &\leq d^{\frac{N-1}{2}}, \quad \text{for } S = 2, \\ \left\| \nu_{S \times \dots \times S}^{(\rho_{d,N})} \right\|_{var} &\leq d^{\frac{S(N-1)}{2}}, \quad \text{for } S \geq 3. \end{aligned} \quad (51)$$

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